Complex curves and non-perturbative effects in $c = 1$ string theory*

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Abstract

We investigate a complex curve in the $c = 1$ string theory which provides a geometric interpretation for different kinds of D-branes. The curve is constructed for a theory perturbed by a tachyon potential using its matrix model formulation. The perturbation removes the degeneracy of the non-perturbed curve and allows to identify its singularities with ZZ branes. Also, using the constructed curve, we find non-perturbative corrections to the free energy and elucidate their CFT origin.

1 Inrtoduction

The $c = 1$ string theory represents an interesting laboratory to study many phenomena inherent to critical string theory (for reviews, see [1, 2, 3]). As it was realized recently, it possesses different kinds of D-branes and its formulation in terms of matrix quantum mechanics (MQM) can be seen as a kind of open/closed string duality [4, 5, 6]. At the same time, this theory is exactly solvable. Due to this it allows to test the ideas and to discover interesting structures which it is difficult to see in more complicated cases.

One of such structures which, although known long ago, appeared in the recent analysis of D-branes in non-critical string theories is a complex curve capturing different aspects of the theory. In non-critical strings it is associated with any closed string background and incorporates information about both closed and open string amplitudes.

However, such a complex curve was constructed and interpreted in terms of D-branes only in the case of $c < 1$ string theories [7, 8]. The $c = 1$ limit of this construction turns out to be degenerate. As a result, several conclusions achieved for $c < 1$ were not evident in the $c = 1$ case. In particular, the D-brane content was not understood although some predictions were made from the matrix model analysis [9].

In this paper we review how one can overcome these difficulties considering the $c = 1$ string theory perturbed by a tachyon potential [10]. Since the CFT technique is not enough powerful to carry out calculations in such situation, we use the MQM formulation to solve the theory and to construct the associated complex curve. Besides, using the resulting curve, we find non-perturbative corrections to the free energy which are related to D-instantons in two-dimensional string theory. Finally, we interpret the matrix model results in the CFT terms. In particular, we show the relation of the CFT complex curve to the matrix model one and identify the origin of the non-perturbative contributions with a subset of ZZ branes.

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2 D-branes and complex curves in minimal string theories

2.1 D-branes in Liouville theory

The recent progress in understanding of non-perturbative effects in non-critical string theories is related with the discovery of conformally invariant boundary conditions (D-branes) in Liouville theory [11, 12]. The latter is defined by the following action:

$$S_L = \int_\Sigma \frac{d^2z}{4\pi}\sqrt{g}\left((\partial\phi)^2 + Q\hat{R}\phi + 4\pi\mu_{_L} e^{2b\phi}\right). \tag{1}$$

The central charge of this CFT is given by

$$c_L = 1 + 6Q^2 \tag{2}$$

and the parameter b is related to Q via the relation

$$Q = b + 1/b. \tag{3}$$

In string theory these parameters are determined by the requirement that the total central charge of matter and the Liouville field is equal to 26. If matter is represented by a minimal (p, q) model with the central charge $c_{p,q} = 1 - 6\frac{(p-q)^2}{pq}$, the relation (2) implies that $b = \sqrt{p/q}$, whereas the coupling to the $c = 1$ matter corresponds to the limit $b \to 1$.

The Liouville theory possesses two types of D-branes, or boundary conditions of Neumann and Dirichlet type. The former, the so called FZZ branes, correspond to the following additional boundary term

$$S_{\rm bnd} = \int_{\partial\Sigma} d\xi\, g^{1/4}\left(\frac{Q\hat{K}}{2\pi}\phi + \mu_{_B} e^{b\phi}\right), \tag{4}$$

and they are parameterized by the boundary cosmological constant $\mu_{_B}$. In fact, it is more convenient to work with another parameter s which is related to $\mu_{_B}$ through the following relation

$$\mu_{_B} = \sqrt{\frac{\mu_{_L}}{\sin(\pi b^2)}}\,\cosh(\pi b s). \tag{5}$$

At the quantum level, the FZZ brane can be characterized by the boundary state which contains information about one-point correlation functions of the bulk operators $V_\alpha = e^{2\alpha\phi}$ on the disk with the boundary condition labeled by s [11]

$$\langle B_s| = \int_0^\infty dP\, U(\alpha(P); s) = \int_0^\infty dP\, \cos(2\pi P s)\Psi(P)\langle P|, \tag{6}$$

where $\alpha(P) = Q/2 - iP$ and

$$\Psi(P) = \left(\pi\mu_{_L}\gamma(b^2)\right)^{-\frac{iP}{b}}\frac{\Gamma(1+2iPb)\Gamma\left(1+2iP/b\right)}{i\pi P}. \tag{7}$$

The Dirichlet boundary conditions, which are also called ZZ branes, appear as non-equivalent quantizations of Lobachevskiy geometry on the world sheet and thus describe branes living at $\phi = \infty$. Similarly to the previous case, they are characterized by a boundary state. It turns out that at the quantum level there is a two-parameter family of consistent boundary conditions. They are referred as (m, n) ZZ branes where m and n run over positive integers. The corresponding boundary states are [12]

$$\langle B_{m,n}| = 2C\int_0^\infty dP\, \sinh(2\pi n P b)\sinh(2\pi m P/b)\Psi(P)\langle P| \tag{8}$$

with C being some numerical constant.

Other correlation functions of bulk and boundary operators on the disk, which are not given in (6) and (8), were found in [13, 14, 15].

2.2 Complex curve of minimal string theories

At first sight the two types of boundary conditions seem to be completely independent. But it is easy to check that they satisfy the following property [5, 14, 16]

$$\langle B_{m,n}| = C\left[\langle B_{s(m,n)}| - \langle B_{s(m,-n)}|\right], \qquad \text{where} \qquad s(m,n) = i\left(\frac{m}{b} + nb\right). \tag{9}$$

Note that the two values of the parameter s, $s(m,n)$ and $s(m,-n)$, correspond to the same value of the boundary cosmological constant $\mu_{_B}$. This hints that the relation (9) realizes a monodromy property of the FZZ boundary state continued analytically to complex values of $\mu_{_B}$ [17]. And indeed, in the case of $c<1$ string theories, a nice geometric interpretation for (9) was found in terms of a complex curve, which also provided a unified description for different kinds of D-branes [7].

The curve comes from two sectors of the theory. Its first origin is the ground ring of closed string vertex operators. In this way the curve encompasses an information about the closed string background. The second origin of the complex curve is the disk partition function with Neumann boundary condition on the Liouville field which is interpreted as the amplitude of an open string ending on the FZZ brane. Let us introduce two variables, x and y, related to the boundary cosmological constant and the FZZ partition function, respectively

$$x = \mu_{_B} \sim \cosh(\pi b s), \qquad y = \frac{\partial Z^{FZZ}}{\partial \mu_{_B}} \sim \cosh(\pi s/b). \tag{10}$$

Then, considered as complex variables, they satisfy some algebraic relation, $F(x,y)=0$, which represents an equation of the complex curve embedded into $\mathbf{C}^2$. From the definition (10), it follows that the FZZ partition function is given by a line integral on the curve of the holomorphic differential ydx, whereas the property (9) ensures that the disk partition function of the (m,n) ZZ brane is evaluated by a similar integral along a closed contour $\gamma_{m,n}$ going from and returning to the point $(x_{m,n} = x(s(m,n)), y_{m,n} = y(s(m,n)))$

$$Z^{FZZ}(\mu_{_B}) = \int_{\mathcal{P}}^{\mu_B} ydx, \qquad Z^{ZZ}_{m,n} = C\oint_{\gamma_{m,n}} ydx.$$

It was shown [7] that the points $(x_{m,n}, y_{m,n})$ coincide with the singularities of the complex curve where it touches itself and forms a "pinched cycle", so that the contours $\gamma_{m,n}$ are non-contractible. Thus, in the $c<1$ case each ZZ brane is associated with a singularity of the complex curve which provides a unified description for both ground ring and FZZ branes.

2.3 The $c=1$ limit of the $c<1$ curve

To generalize the above picture to the case of the $c=1$ string theory, let us take the limit $b \to 1$ in equations (10) defining the complex curve. It is known that this limit is singular and requires the following renormalization of couplings

$$\mu_{_{c=1}} = \lim_{b\to 1}\left[\pi(1-b^2)\mu_{_L}\right], \quad \mu_{_{B,c=1}} = \lim_{b\to 1}\left[\pi(1-b^2)\mu_{_B}\right]. \tag{11}$$

However, it is easy to see that it is not enough because even after the renormalization, for example, the FZZ partition function remains singular. However, the singular term is "non-universal" from the point of view of the $c=1$ theory since it is polynomial in $\mu_{_B}$. Thus, this term should be subtracted so that we define the renormalized one-point disk correlation function with FZZ boundary condition as follows

$$w(s) \equiv \lim_{b\to 1}\left(\frac{\partial_{\mu_B} Z^{FZZ}}{\pi(1-b^2)} + \frac{4}{\pi} Z^{c=1}_{\text{Dir}}\,\mu_{_B}\right), \tag{12}$$

where $Z_{\rm Dir}^{c=1}$ is the disk partition function in the $c = 1$ CFT. Then the simple calculation leads to the following result [10]

$$\mu_{B,c=1} = \sqrt{\mu_{c=1}}\cosh(\pi s), \qquad w(s) = -D\sqrt{\mu_{c=1}}\,\pi s\sinh(\pi s), \tag{13}$$

where D is some constant. This gives a parametric representation of the limiting curve coming from the FZZ partition function. Contrary to the previous case, all singularities of this curve collapse just to two points. Indeed, the $c = 1$ limit of (9) reads

$$s(m,n) = i(m+n), \qquad \Big(\mu_{B,c=1}(m,n), w(m,n)\Big) = \left((-1)^{m+n}\sqrt{\mu_{c=1}}, 0\right). \tag{14}$$

As a result, the curve is degenerate and does not allow to make unambiguous identification between the singularities and the ZZ branes. Also, as we will see, it differs from the curve associated with the ground ring of the $c = 1$ string theory. To understand these issues, to construct a non-degenerate curve and to find some non-perturbative effects, we turn now to the matrix model formulation of the two-dimensional string theory.

3 Complex curve of the perturbed MQM

3.1 Complex curve from the profile of the Fermi sea

The matrix model in question is the double scaled matrix quantum mechanics. More precisely, we are interested in its singlet sector where it can be reduced to a system a free fermions in the inverted oscillator potential. At the quasiclassical level the system of free fermions can be completely characterized by the shape of the Fermi sea in the phase space formed by the matrix eigenvalue x, playing the role of the fermion coordinate, and its conjugated momentum p.

The ground state, corresponding to the simplest linear dilaton background on the string side, is described by the Fermi sea of the hyperbolic shape, $\frac{1}{2}(x^2 - p^2) = \mu$, where μ is the Fermi level. Let us parameterize it in the way similar to (13)

$$x(\tau) = \sqrt{2\mu}\cosh(\tau), \qquad y(\tau) = \sqrt{2\mu}\sinh(\tau). \tag{15}$$

Then we can continue the parameter τ to the complex plane and view the equation for the profile of the Fermi sea as an equation for the complex curve associated with the given solution of MQM. This curve differs from the curve (13) obtained from the FZZ partition function. In fact, it is this curve that describes the ground ring in the $c = 1$ case. It is easy to see if one introduces the so called light-cone coordinates in the phase space

$$x_\pm = \frac{x \pm p}{2}. \tag{16}$$

In terms of these coordinates the equation for the curve takes the simple form $x_+x_- = \mu$ and coincides with the equation found by Witten [18] provided one identifies $x_\pm$ with the generators of the ground ring.

But the complex curve (15) is even more degenerate than the one from (13) since it covers itself infinitely many times. We expect that the degeneracy will disappear after we perturb the theory by some relevant operators. The simplest way to do this is to introduce a tachyon potential characterized by couplings λ_n. Thus, we are going to consider the theory with the following action

$$S_{c=1} = \int_\Sigma \frac{d^2 z}{4\pi}\left[(\partial X)^2 + (\partial\phi)^2 + 2\hat{\mathcal{R}}\phi + \mu_{_L} e^{2\phi} + \sum_{n\geq 1}\lambda_n\, e^{(2-\frac{n}{R})\phi}\cos\left(\tfrac{nX}{R}\right)\right]. \tag{17}$$

In MQM such a closed string background is described by a time-dependent Fermi sea. Its profile is determined as a consistent solution of the following two equations [19]

$$x_+ x_- = \frac{1}{R}\sum_{k\geq 1} k\lambda_k\, x_\pm^{k/R} + \mu + \frac{1}{R}\sum_{k\geq 1} v_k(\mu,\lambda)\, x_\pm^{-k/R}, \tag{18}$$

where the coefficients v_k contain information about one-point correlation functions and are fixed by the compatibility condition. The solution is given in the parametric form [19]

$$x_\pm(\tau) = e^{-\frac{1}{2R}\partial_\mu^2 \mathcal{F}_0}\, e^{\pm\tau}\left(1+\sum_{k\geq 1} a_k(\mu,\lambda)\, e^{\mp k\tau/R}\right), \tag{19}$$

where the coefficients a_k can be found explicitly and $\mathcal{F}_0$ is the free energy on the sphere.

As we learned above, the profile of the Fermi sea defines the MQM complex curve. Thus, the solution (19) is what we are looking for! If $\tau \in \mathbf{C}$, $(x_+(\tau), x_-(\tau))$ defines an embedding of the complex curve into $\mathbf{C}^2$ and the parameter τ is its uniformization parameter. It is easy to check that for non-vanishing couplings the curve is not degenerate anymore. It possesses singularities given by a one-parameter set of points $\tau_n = i\theta_n \in i\mathbf{R}$ satisfying $x_\pm(\tau_n) = x_\pm(-\tau_n)$. Thus, θ_n can be found from the following algebraic equation

$$\sin\theta_n = \sum_{k\geq 1} a_k \sin\left(\left(\tfrac{k}{R}-1\right)\theta_n\right), \tag{20}$$

where the solutions are ordered in such way that $\theta_n = \pi n + O(\lambda)$. For a generic value of the parameter R all of them are different, whereas for rational R only a finite set survives. A more detailed analysis of the complex curve can be found in [10].

3.2 Non-perturbative effects from the MQM complex curve

Similarly to the case of $c < 1$, we expect that the complex curve constructed here contains information about D-branes and therefore it is able to describe some of the non-perturbative effects. We will be interested in a particular kind of such effects which are given by non-perturbative corrections to the closed string partition function. This quantity is represented in the matrix model by the free energy and is a sum of the perturbative and non-perturbative parts

$$\mathcal{F} = \mathcal{F}_{\rm pert} + \mathcal{F}_{\rm non-pert} = \sum c_n\, g_s^{2n-2} + \sum a_n\, e^{-d_n/g_s} \tag{21}$$

The first one is the series in the string coupling and gives closed string amplitudes, whereas the second part has an exponential form and, as it was shown for critical string theories [20] and latter in the leading order for non-critical strings [21, 8], describes amplitudes of open strings attached to D-instantons.

The analysis of the fermion system in the presence of arbitrary tachyon perturbation introduced in (17) allows to calculate the non-perturbative coefficients d_n and a_n explicitly. We refer to [22] for details of the calculation. The result reads

$$d_n = i\int_{\gamma_n} x_-\, dx_+, \tag{22}$$

$$a_n \sim \left[\sin^2\frac{\theta_n}{R}\left(\left.\frac{\partial x_+}{\partial\tau}\right|_{-i\theta_n}\left.\frac{\partial x_-}{\partial\tau}\right|_{i\theta_n} - \left.\frac{\partial x_+}{\partial\tau}\right|_{i\theta_n}\left.\frac{\partial x_-}{\partial\tau}\right|_{-i\theta_n}\right)\right]^{-1/2}, \tag{23}$$

where θ_n were defined to parameterize the singularities of the complex curve in (20) and the contours γ_n are the images of the intervals $(i\theta_n, -i\theta_n)$ under the map (19). One observes that each non-perturbative correction is associated with one of the singularities. This points toward the relation between the singularities and the localized ZZ branes which was found before for $c < 1$ string theories.

4 CFT interpretation of the MQM results

To establish connection of the previous results with the CFT description, one should know correlation functions in the perturbed theory (17). Unfortunately, it seems to be a hopeless problem at the present moment. Therefore, our method is to expand the matrix model results in the couplings and to compare the first two terms of the expansion. For example, if only the first coupling λ_1 is non-vanishing, these terms should correspond to the correlation functions of the cosmological constant and the Sine-Liouville operators, respectively, which can be calculated taking the $b=1$ limit of the results of [11, 12]

$$Z(\lambda_{_L}) = \lim_{b\to 1}\left(Z_{\rm Dir}^{c=1}\int d\mu\, \langle V_b\rangle + \lambda_{_L}\left\langle \cos\left(\tfrac{X}{R}\right)\right\rangle_{\rm Dir}\left\langle V_{b-\frac{1}{2R}}\right\rangle + \ \dots \ \begin{matrix}\text{multi-point}\\ \text{correlators}\end{matrix}\ \dots\right). \tag{24}$$

If these terms are the same in MQM and CFT, we claim that the coincidence remains to be true for the whole series so that the two formulations give the same result for the quantity under consideration at any values of the couplings. Then the other terms of the expansion provide a matrix model prediction for multi-point correlation functions.

Following this approach and taking into account the relation between the matrix model and CFT couplings, one can show [10] that the MQM curve is exactly the same as a CFT curve provided the latter is defined through the FZZ partition function as follows

$$x(s) = \sqrt{2\mu}\,\cosh(\pi s) \sim \mu_{_{B,c=1}}, \qquad p(s) = -\frac{C}{2i}\left[\partial_x Z^{FZZ}(s+i) - \partial_x Z^{FZZ}(s-i)\right]. \tag{25}$$

This result is quite natural because it is well known that $\partial_x Z^{FZZ}$ is proportional to the matrix model resolvent, whereas the momentum p measures the "width" of the Fermi sea, *i.e.*, the density of eigenvalues. Thus, the identification (25) is nothing else but the standard relation between the density and the resolvent. Note also that the first equation in (25) allows to find the relation between the uniformization parameter τ and the CFT parameter s. They coincide (up to the factor π) only in the non-perturbed case, whereas in general one is a complicated function of the other.

As an example of a matrix model prediction, we give the two-point correlation function on the disk of the Sine-Liouville operator with FZZ boundary conditions on the Liouville and Dirichlet conditions on the matter field, which follows from the second order term in the expansion of (25) [10]

$$\left\langle\left(\int d^2\sigma e^{\left(2-\frac{1}{R}\right)\phi}\cos\tfrac{X}{R}\right)^2\right\rangle = -\frac{\pi\Gamma^2\left(1-\frac{1}{R}\right)\lambda_{_L}^2\mu_{c=1}^{\frac{1}{R}-1}}{2^{5/4}\sqrt{R}\,\Gamma^2\left(\frac{1}{R}\right)}\left(s\coth(\pi s) + \frac{\sinh\left(\left(\frac{2}{R}-1\right)\pi s\right)}{\sin\frac{2\pi}{R}\sinh(\pi s)}\right). \tag{26}$$

To complete the identification between the MQM and CFT structures, it remains to show that the singularities of the complex curve labeled by θ_n from (20) are in one to one correspondence with ZZ branes. Since each singularity of the curve gives rise to a non-perturbative correction to the free energy (21), the coefficients a_n and d_n should have an interpretation in terms of correlation functions of open strings ending on the ZZ branes. Indeed, it is easy to check [21, 9, 10] (again in the first two orders in the coupling constants) that the leading correction d_n coincides with the partition function on the disk with $(n,1)$ ZZ boundary conditions

$$-\,d_n = i\int_{\gamma_n} p\, dx = Z_{n,1}^{ZZ}. \tag{27}$$

This means that the nth singularity of the complex curve is associated with the $(n,1)$ ZZ brane and only this set of branes survives in the $c=1$ limit [9].

However, the interpretation of the subleading non-perturbative contribution a_n is still lacking. It should be related to the annulus amplitude $Z_{\rm annulus}^{ZZ}(n,1;n,1)$ between two $(n,1)$ branes, but

the known expression for this quantity diverges [23]. Nevertheless, it can be obtained from two-point correlators of a Gaussian field arising after bosonization of the chiral fermions [22]. But the connection of this field to either ZZ or FZZ branes is not clear to us.

5 Discussion

We presented the construction of the complex curve of the $c = 1$ string theory. The formulation in terms of matrix quantum mechanics allowed to find the curve for the theory perturbed by a tachyon potential. In contrast to the non-perturbed case, the resulting curve is not degenerate and its singularities are associated with the set of $(n, 1)$ ZZ branes. As in the $c < 1$ case, the disk partition functions of these branes are given by contour integrals on the curve passing through the singularities.

An important distinction with $c < 1$ string theories is that in the $c = 1$ case the complex curve of the ground ring coincides with the curve defined by the density of matrix eigenvalues, whereas for $c < 1$ it is associated with the matrix model resolvent. The usual relation between the density and the resolvent allows to obtain one curve from the other. One can show that this transformation does not affect the singularities and their relation to D-branes [10].

The knowledge of the results for the perturbed theory allows to predict many correlation functions with FZZ and ZZ boundary conditions. It is enough to expand the results in the couplings and make a careful identification of all quantities. We gave here just one example — the two-point correlator of the Sine–Liouville operator. However, up to now not all found quantities have a known counterpart in the CFT formulation. An important problem which remains open is to find such an interpretation for the subleading non-perturbative correction. Thus, the relation between MQM and the continuum formulation beyond the leading order remains still mysterious.

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